The Group of Extensions in $\operatorname{rep} Q$

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Throughout, we will assume Q is a given acyclic quiver, and we will work in the category of representations of Q over a given base field k. In §1, we define $\mathcal{E}(M, N)$, the extensions of M by N for two representations M, N, and define a binary operation on $\mathcal{E}(M, N)$. In §2, we show that this operation makes $\mathcal{E}(M, N)$ an abelian group. In §3, we describe an isomorphism of $\mathcal{E}(M, N)$ with $\operatorname{Ext}^1(M, N)$.

1 Definition of $\mathcal{E}(M, N)$

Much of this discussion of extensions parallels extensions in the category of groups or R-modules. For some discussion of the R-module version, see Weibel's *Introduction to Homological Algebra* [2].

Definition 1.1. *Let $M, N \in \operatorname{rep} Q$. An **extension** ζ of M by N is a short exact sequence of the form*

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

Definition 1.2. *Two extensions ζ, ζ' of M by N are **equivalent** if there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \operatorname{Id}_N} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \operatorname{Id}_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Note that by the Five Lemma, any such ϕ is an isomorphism.

Definition 1.3. *The **group of extensions** $\mathcal{E}(M, N)$ of M by N is the set of equivalence classes of extensions of M by N. (We haven't yet defined a group structure on this set, but we will.)*

Our first objective is to define an abelian group structure on $\mathcal{E}(M, N)$. Our second objective is to show that $\mathcal{E}(M, N) \cong \operatorname{Ext}^1(M, N)$ as abelian groups, after defining $\operatorname{Ext}^1(M, N)$.

First, we define a binary operation on extensions. Then we will show that it is well defined on equivalence classes of extensions.

Definition 1.4. *Let $M, N \in \operatorname{rep} Q$, and let ζ, ζ' be the following extensions of M by N.*

$$\zeta \qquad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

$$\zeta' \qquad 0 \longrightarrow N \xrightarrow{f'} E' \xrightarrow{g'} M \longrightarrow 0$$

Define

$$E'' = \{(x, x') \in E \oplus E' : g(x) = g'(x')\}$$
$$D'' = \{(f(n), -f'(n)) \in E \oplus E' : n \in N\}$$

Then define $F = E''/D''$. Finally, the extension $\zeta + \zeta'$ is defined to be

$$0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0$$

where $f''(n) = \overline{\big(f(n), 0\big)}$ and $g''(\overline{x, x'}) = g(x)$.

This addition is called the **Baer sum**, at least in the context of R-modules.

Lemma 1.1. *The definition above makes sense. More specifically,*

1. *E'' and D'' are representations of Q, and D'' is a subrepresentation of E''.*
2. *f'' and g'' are well defined and are morphisms in $\operatorname{rep} Q$.*
3. *The sequence involving F is exact.*

Proof. (1) E'' is a representation of Q by Exercise 1.8 in [1]. D'' is a subrepresentation of $E \oplus E'$ by Exercise 1.9 in [1]. Also, $D'' \subset E''$, since

$$g(f(n)) = 0 = g'(-f'(n) \qquad \forall n \in N$$

(2) It is clear that f'' is well defined and is a morphism. We check that g'' is well defined by showing that it vanishes on D''.

$$g''\overline{(f(n), -f'(n))} = gf(n) = 0$$

It is clear that g'' is a morphism, since g is a morphism.

(3) We check that the sequence involving F is exact. First, we show injectivity of f''. If $n \in \ker f''$, then there exists $n' \in N$ such that

$$f''(n) = (f(n), 0) = (f(n'), -f'(n')) \implies 0 = -f'(n')$$

which implies $n' = 0$ by injectivity of f'. Then $f(n') = 0$ so $f(n) = 0$ as well, so $n = 0$ by injectivity of f. Thus f'' is injective. Now we show g'' is surjective. Let $m \in M$. By surjectivity of g, g', there exist $x \in E, x' \in E'$ so that $g(x) = g'(x') = m$. Then

$$g''\overline{(x, x')} = g(x) = m$$

so g'' is surjective. Finally, we show that $\ker g'' = \operatorname{im} f''$. It is easy to see that $\operatorname{im} f'' \subset \ker g''$, since

$$g''f''(n) = g''\overline{(f(n), 0)} = gf(n) = 0$$

We need to check that $\ker g'' \subset \operatorname{im} f''$. Let $\overline{(x, x')} \in \ker g''$, so $0 = g(x) = g'(x')$. By exactness of ζ, ζ', $x \in \operatorname{im} f$ and $x' \in \operatorname{im} f'$, so there exist $n, n' \in N$ such that $f(n) = x$ and $f'(n') = x'$. Then

$$\begin{aligned} f''(n + n') &= \overline{(f(n + n'), 0)} \\ &= \overline{(f(n) + f(n'), 0)} \\ &= \overline{(f(n) + f(n'), 0)} + \overline{(f(-n'), -f'(-n'))} \\ &= \overline{(f(n), -f'(n'))} \\ &= \overline{(x, x')} \end{aligned}$$

thus $\ker g'' \subset \operatorname{im} f''$. □

With this lemma in hand, we know that our addition is well defined on exact sequences. Now we need to check that it induces a well defined addition on $\mathcal{E}(M, N)$.

Definition 1.5. *Let $[\zeta], [\zeta']$ be equivalence classes of extensions in $\mathcal{E}(M, N)$. We define addition in $\mathcal{E}(M, N)$ by*

$$[\zeta] + [\zeta'] = [\zeta + \zeta']$$

Lemma 1.2. *This addition on $\mathcal{E}(M, N)$ is well defined.*

Proof. We need to show that if $[\gamma] = [\zeta]$ and $[\gamma'] = [\zeta']$, then $[\gamma + \gamma'] = [\zeta] + [\zeta']$. Let $\zeta, \zeta', \gamma, \gamma', \zeta + \zeta', \gamma + \gamma'$ be the following extensions.

$$\zeta \qquad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

$$\zeta' \qquad 0 \longrightarrow N \xrightarrow{f'} E' \xrightarrow{g'} M \longrightarrow 0$$

$$\zeta + \zeta' \qquad 0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0$$

$$\gamma \qquad 0 \longrightarrow N \xrightarrow{h} S \xrightarrow{j} M \longrightarrow 0$$

$$\gamma' \qquad 0 \longrightarrow N \xrightarrow{h'} S' \xrightarrow{j'} M \longrightarrow 0$$

$$\gamma + \gamma' \qquad 0 \longrightarrow N \xrightarrow{h''} T \xrightarrow{j''} M \longrightarrow 0$$

where $F = E''/D''$ and $T = S''/R''$. Because $[\gamma] = [\zeta]$ and $[\gamma'] = [\zeta']$, there is are isomorphisms $\phi : E \to S$ and $\phi' : E' \to S'$ making the following diagrams commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{h} & S & \xrightarrow{j} & M & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \phi'} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{h'} & S' & \xrightarrow{j'} & M & \longrightarrow & 0 \end{array}$$

Then we have an isomorphism $\phi \oplus \phi' : E \oplus E' \to S \oplus S'$ given by $(x, x') \mapsto (\phi(x), \phi'(x'))$. We claim that $\phi \oplus \phi'$ induces an isomorphism $F \to T$ giving an equivalence $[\zeta + \zeta'] = [\gamma + \gamma']$.

First, we claim that $\phi \oplus \phi'|_{E''} : E'' \to S \oplus S'$ has image contained in S''. This follows from the right side commutative squares. For $(x, x') \in E''$, we have $g(x) = g'(x')$, so

$$\phi \oplus \phi'(x, x') = (\phi(x), \phi'(x')) \in S'' \text{ because } j\phi(x) = g(x) = g'(x') = j'\phi'(x')$$

We also claim S'' is contained in the image. For $(y, y') \in S''$, we have $j(y) = j'(y')$, so $(\phi^{-1}(y), (\phi')^{-1}(y')) \in E''$ because $g\phi^{-1}(y) = j(y) = j'(y') = g'(\phi')^{-1}(y')$. Thus

$$\phi \oplus \phi'(\phi^{-1}(y), (\phi')^{-1}(y')) = (y, y')$$

so S'' is the image. Now we claim that $\phi \oplus \phi'|_{D''} : D'' \to S \oplus S''$ has image R''. Containment and surjection follow from left side commutative squares, as seen below.

$$\phi \oplus \phi'(f(n), -f'(n)) = (\phi f(n), -\phi' f'(n)) = (h(n), -h'(n)) \in R''$$

So $\phi \oplus \phi'$ restricts to isomorphisms $E'' \to S''$ and $D'' \to R''$. Thus $\phi \oplus \phi'$ induces an isomorphism $E''/D'' \to S''/R''$, that is, $F \to T$, making the following diagram commute.

$$\begin{array}{cccccccccc} \zeta + \zeta' & \qquad 0 & \longrightarrow & N & \xrightarrow{f''} & F & \xrightarrow{g''} & M & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle \overline{\phi \oplus \phi'}} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\ \gamma + \gamma' & \qquad 0 & \longrightarrow & N & \xrightarrow{h''} & T & \xrightarrow{j''} & M & \longrightarrow & 0 \end{array}$$

Thus $[\zeta + \zeta'] = [\gamma + \gamma']$. □

2 Verifying Group Axioms

Proposition 2.1. *$\mathcal{E}(M, N)$ is an abelian group with this addition.*

We break this into several separate propositions, so that the reader can easily find the proof of a particular property.

Proposition 2.2. *The split extension is an additive identity in $\mathcal{E}(M, N)$.*

Proof. First, we claim that the equivalence class of the sequence $[\alpha]$, depicted below,

$$0 \longrightarrow N \xrightarrow{\iota} N \oplus M \xrightarrow{\pi} M \longrightarrow 0$$

acts as an addtive identity in $\mathcal{E}(M,N)$. Let $[\zeta] \in \mathcal{E}(M,N)$ with representative

$$0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

Then we set

$$\begin{aligned} E'' &= \{(e,(n,m)) \in E \oplus (N \oplus M) : g(e) = \pi(n,m)\} \\ &= \{(e,(n,m)) \in E \oplus (N \oplus M) : g(e) = m\} \\ D'' &= \{(f(n), -\iota(n)) \in E \oplus (N \oplus M) : n \in N\} \\ &= \{(f(n),(-n,0)) \in E \oplus (N \oplus M)\} \\ F &= E''/D'' \end{aligned}$$

Then $[\zeta + \alpha]$ is represented by

$$0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0$$

where $f''(n) = \overline{(f(n),(0,0))} = \overline{(f(n),(0,0))}$ and $g''\overline{(e,(n,m))} = g(e)$. We claim that $[\zeta+\alpha] = [\zeta]$. To show this equivalence, we exhibit an explicit equivalence of extensions. Define $\phi : E \to F$ by $e \mapsto \overline{(e,(0,g(e)))}$ and $\psi : F \to E$ by $\overline{(e,(n,m))} \mapsto e + f(n)$. It is straightforward to see that ϕ is well defined, maps into F, and is a morphism. We check that ψ is well defined by checking that it vanishes on D''.

$$\psi\overline{(f(n),(-n,0))} = f(n) + f(-n) = 0$$

It is clear that ψ maps into E and is a morphism. Now we show that ϕ, ψ are inverse.

$$\begin{aligned} \psi\phi(e) &= \psi\overline{(e,(0,g(e)))} = e + f(0) = e \\ \phi\psi\overline{(e,(n,m))} &= \phi(e + f(n)) = \overline{(e + f(n),(0,g(e+f(n))))} \end{aligned}$$

Finally, we check that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \big\downarrow \text{Id} & & \big\downarrow \phi & & \big\downarrow \text{Id} & & \\ 0 & \longrightarrow & N & \xrightarrow{f''} & F & \xrightarrow{g''} & M & \longrightarrow & 0 \end{array}$$

$$\begin{aligned} \phi f(n) &= \overline{(f(n),(0,0))} = f''(n) \\ g''\phi(e) &= g''\overline{(e,(0,g(e)))} = g(e) \end{aligned}$$

Thus $[\zeta + \alpha] = [\zeta]$, so $[\alpha]$ is an identity in $\mathcal{E}(M,N)$. □

Proposition 2.3. *Addition in $\mathcal{E}(M,N)$ is associative.*

Proof. Let ζ_i for $i = 1,2,3$ be extensions of M by N.

$$\zeta_i \qquad 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$$

Let

$$\begin{aligned} E_{ij} &= \{(x_i, x_j) \in E_i \oplus E_j : g_i(x_i) = g_j(x_j)\} \\ D_{ij} &= \{(f_i(n), -f_j(n)) : n \in N\} \\ F_{ij} &= E_{ij}/D_{ij} \end{aligned}$$

and define $f_{ij} : N \to F_{ij}$ by $f_{ij}(n) = \overline{(f_i(n), 0)}$ and $g_{ij} : F_{ij} \to M$ by $g_{ij}\overline{(x_i, x_j)} = g_i(x_i)$. That is, $\zeta_i + \zeta_j$ is represented by

$$0 \longrightarrow N \xrightarrow{f_{ij}} F_{ij} \xrightarrow{g_{ij}} M \longrightarrow 0$$

Then set

$$\begin{aligned} E_{(ij)k} &= \left\{(\overline{(x_i, x_j)}, x_k) \in F_{ij} \oplus E_k : g_{ij}\overline{(x_i, x_j)} = g_k(x_k)\right\} \\ D_{(ij)k} &= \{(f_{ij}(n), f_k(n)) : n \in N\} \\ F_{(ij)k} &= E_{(ij)k}/D_{(ij)k} \\ E_{i(jk)} &= \left\{(x_i, \overline{(x_j, x_k)}) \in E_i \oplus F_{jk} : g_i(x_i) = g_{jk}\overline{(x_j, x_k)}\right\} \\ D_{i(jk)} &= \{(f_i(n), f_{jk}(n)) : n \in N\} \\ F_{i(jk)} &= E_{i(jk)}/D_{i(jk)} \end{aligned}$$

and let $f_{(ij)k}, g_{(ij)k}$ and $f_{i(jk)}, g_{i(jk)}$ so that $(\zeta_i + \zeta_j) + \zeta_k$ and $\zeta_i + (\zeta_j + \zeta_k)$ are respectively represented by

$$0 \longrightarrow N \xrightarrow{f_{(ij)k}} F_{(ij)k} \xrightarrow{g_{(ij)k}} M \longrightarrow 0$$

$$0 \longrightarrow N \xrightarrow{f_{i(jk)}} F_{i(jk)} \xrightarrow{g_{i(jk)}} M \longrightarrow 0$$

We care about the case $i = 1, j = 2, k = 3$. We define $\Psi : E_{(12)3} \to E_{1(23)}$ by

$$\Psi\left(\overline{(x_1, x_2)}, x_3\right) = \left(x_1, \overline{(x_2, x_3)}\right)$$

First, we need to check that this is well defined; for this it is sufficient to check that Ψ vanishes on the zero element of $E_{(12)3}$. We can represent the zero element of $E_{(12)3}$ by $(\overline{(0,0)}, 0)$, which clearly goes to the zero element of $E_{1(23)}$ under Ψ, so it is well defined.

We also need to check that the image is contained in $E_{1(23)}$. For $\left(\overline{(x_1, x_2)}, x_3\right) \in E_{(12)3}$ we have $g_{12}\overline{(x_1, x_2)} = g_3(x_3)$, so $g_1(x_1) = g_2(x_2) = g_3(x_3)$ (because $(x_1, x_2) \in E_{12}$). Thus $g_1(x_1) = g_{23}\overline{(x_2, x_3)}$, so the image is contained in $E_{1(23)}$ as desired.

Now we claim that Ψ maps $D_{(12)3}$ to $D_{1(23)}$. For $n \in N$,

$$\Psi\left(f_{12}(n), f_3(n)\right) = \Psi\left(\overline{(f_1(n), 0)}, f_3(n)\right) = \Psi\left(\overline{(0, f_2(n))}, f_3(n)\right) = \left(0, \overline{(f_2(n), f_3(n))}\right) \in D_{1(23)}$$

Thus Ψ induces a morphism $F_{(12)3} \to F_{1(23)}$. Finally, we need to check that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f_{(ij)k}} & F_{(ij)k} & \xrightarrow{g_{(ij)k}} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{f_{i(jk)}} & F_{i(jk)} & \xrightarrow{g_{i(jk)}} & M & \longrightarrow & 0 \end{array}$$

Note that $f_{(12)3}(n) = \overline{(f_{12}(n), 0)}$ and $f_{1(23)}(n) = \overline{\left(f_1(n), \overline{(0,0)}\right)}$ and $g_{(12)3}\overline{\left(\overline{(x_1, x_2)}, x_3\right)} = g_{12}\overline{(x_1, x_2)} = g_1(x_1)$ and $g_{1(23)}\overline{\left(x_1, \overline{(x_2, x_3)}\right)} = g_1(x_1)$.

$$\Psi f_{(12)3}(n) = \Psi\overline{(f_{12}(n), 0)} = \Psi\overline{\left(\overline{(f_1(n), 0)}, 0\right)} = \overline{\left(f_1(n), \overline{(0,0)}\right)} = f_{1(23)}(n)$$
$$g_{1(23)}\Psi\overline{\left(\overline{(x_1, x_2)}, x_3\right)} = g_{1(23)}\left(x_1, \overline{(x_2, x_3)}\right) = g_1(x_1) = g_{(12)3}\overline{\left(\overline{(x_1, x_2)}, x_3\right)}$$

Thus the diagram commutes and Ψ is an equivalence of extensions. (Note that by the Five Lemma, we any morphism making this commute is an isomorphism.) □

Proposition 2.4. *If $[\zeta] \in \mathcal{E}(M, N)$, there is an extension $-\zeta$ so that $[\zeta] + [-\zeta] = [0]$.*

Proof. Let ζ be the extension

$$0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

Then we have another extension, which we call $-\zeta$,

$$0 \longrightarrow N \xrightarrow{-f} E \xrightarrow{g} M \longrightarrow 0$$

We claim that $[\zeta] + [-\zeta] = [0]$, that is, $\zeta + (-\zeta)$ is equivalent to the split extension. Let's describe $\zeta + (-\zeta)$. It is

$$0 \longrightarrow N \xrightarrow{f''} F \xrightarrow{g''} M \longrightarrow 0$$

where

$$\begin{aligned} E'' &= \{(x, x') \in E \oplus E' : g(x) = g(x')\} \\ D'' &= \{(f(n), f(n)) : n \in N\} \\ F &= E''/D'' \end{aligned}$$

and $f''(n) = \overline{(f(n), 0)}$ and $g''\overline{(x, x')} = g(x) = g(x')$. We define a morphism $\phi : N \oplus M \to F$ as follows. For $m \in M$, there exists $x \in E$ so that $g(x) = m$ by surjectivity of g. We define $\phi(n, m) = \overline{(f(n) + x, x)}$. We need to check that this is well defined. Suppose x, x' are two different lifts of m. Then $x - x' \in \ker g$, so there exists $n \in N$ with $f(n') = x - x'$, so for $n \in N$, we have

$$(f(n) + x, x) - (f(n) + x', x') = (x - x', x - x') = (f(n'), f(n')) \in D''$$

which implies that $\overline{f(n) + x, x)} = \overline{(f(n) + x', x')}$. Thus ϕ is well defined. We verify that the diagram below commutes, and thus ϕ is an isomorphism, and we have $[\zeta + (-\zeta)] = [0]$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\iota} & N \oplus M & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & N & \xrightarrow{f''} & F & \xrightarrow{g''} & M & \longrightarrow & 0
\end{array}$$

$$\begin{aligned}
\phi\iota(n) &= \phi(n,0) = \overline{(f(n),0)} = f''(n) \\
g''\phi(n,m) &= g''\overline{(f(n)+x,x)} = g(f(n)+x) = g(x) = m = \pi(m)
\end{aligned}$$

□

Proposition 2.5. *Addition in $\mathcal{E}(M,N)$ is commutative.*

Proof. Let ζ_i for $i = 1, 2$ be extensions of M by N.

$$\zeta_i \qquad 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$$

Let

$$\begin{aligned}
E_{ij} &= \{(x_i, x_j) \in E_i \oplus E_j : g_i(x_i) = g_j(x_j)\} \\
D_{ij} &= \{(f_i(n), -f_j(n)) : n \in N\} \\
F_{ij} &= E_{ij}/D_{ij}
\end{aligned}$$

and define $f_{ij} : N \to F_{ij}$ by $f_{ij}(n) = \overline{(f_i(n), 0)}$ and $g_{ij} : F_{ij} \to M$ by $g_{ij}\overline{(x_i, x_j)} = g_i(x_i)$. That is, $\zeta_i + \zeta_j$ is represented by

$$0 \longrightarrow N \xrightarrow{f_{ij}} F_{ij} \xrightarrow{g_{ij}} M \longrightarrow 0$$

We have the obvious isomorphism $\Psi : E_{12} \to E_{21}$ given by $(x_1, x_2) \mapsto (x_2, x_1)$. Ψ restricts to an isomorphism $D_{12} \to D_{21}$, because

$$\Psi\Big(f_1(n), -f_2(n)\Big) = \Big(-f_2(n), f_1(n)\Big) = \Big(f_2(-n), -f_1(-n)\Big)$$

Thus Ψ induces an isomorphism $F_{12} \to F_{21}$, and we verify that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{f_{12}} & F_{12} & \xrightarrow{g_{12}} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & N & \xrightarrow{f_{21}} & F_{21} & \xrightarrow{g_{21}} & M & \longrightarrow & 0
\end{array}$$

$$\begin{aligned}
\Psi f_{12}(n) &= \Psi\overline{(f_1(n),0)} = \overline{(0, f_1(n))} = \overline{(0, f_1(n))} + \overline{(f_2(n), -f_1(n))} = \overline{(f_2(n), 0)} = f_{21}(n) \\
g_{21}\Psi\overline{(x_1, x_2)} &= g_{21}\overline{(x_2, x_1)} = g_2(x_2) = g_1(x_1) = g_{12}\overline{(x_1, x_2)}
\end{aligned}$$

□

This completes the proof that $\mathcal{E}(M,N)$ is an abelian group.

3 Isomorphism $\mathcal{E}(M,N) \cong \operatorname{Ext}^1(M,N)$

Now that we know that $\mathcal{E}(M,N)$ is an abelian group, we can describe it's relationship with the functor Ext^1. First we recall the definition of Ext^1. Remember that every representation of Q has a two-term projective resolution.

Definition 3.1. *Let $M \in \operatorname{rep} Q$. Let*

$$0 \longrightarrow P_1 \xrightarrow{f} P_2 \xrightarrow{g} M \longrightarrow 0$$

be a projective resolution of M. Then for $N \in \operatorname{rep} Q$, we define $\operatorname{Ext}^1(M,N)$ as the cokernel of f^* in the following sequence.

$$0 \longrightarrow \operatorname{Hom}(M,N) \xrightarrow{g^*} \operatorname{Hom}(P_0,N) \xrightarrow{f^*} \operatorname{Hom}(P_1,N)$$

That is, $\operatorname{Ext}^1(M,N) := \operatorname{Hom}(P_1,N)/\operatorname{im} f^*$. In particular, the following sequence is exact.

$$0 \longrightarrow \operatorname{Hom}(M,N) \xrightarrow{g^*} \operatorname{Hom}(P_0,N) \xrightarrow{f^*} \operatorname{Hom}(P_1,N) \longrightarrow \operatorname{Ext}^1(M,N) \longrightarrow 0$$

Note: It is not clear from this definition why $\operatorname{Ext}^1(M,N)$ does not depend on the choice of projective resolution. However, there are "standard" results in homological algebra that it does not. That is, $\operatorname{Ext}^1(M,N)$ depends on only M and N.

Note that $\operatorname{Ext}^1(M,N)$ is a k-vector space, so it is also an abelian group. Now we will show that it is isomorphic to $\mathcal{E}(M,N)$ as an abelian group.

Definition 3.2. *Fix a projective resolution $\mathcal{P}$ of M.*

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

Let $[\zeta] \in \mathcal{E}(M,N)$ with representative short exact sequnce ζ.

$$0 \longrightarrow N \xrightarrow{s} E \xrightarrow{t} M \longrightarrow 0$$

Since P_0 is projective and t is surjective, there exists $a : P_0 \to E$ making the following diagram commute (by the universal property of projectives).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0
\end{array}$$

By commutativity of this diagram, $taf = gf = 0$, that is, $af : P_0 \to \ker t = \operatorname{im} s$. Since $s : N \to \operatorname{im} s$ is surjective and P_1 is projective, again using the universal property of projectives, there is $b : P_1 \to N$ making the following diagram commute.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0
\end{array}$$

Recall that $\operatorname{Ext}^1(M,N) = \operatorname{Hom}(P_1,N)/\operatorname{im} f^*$, *so* b *is a representative of some class* $\overline{b} \in \operatorname{Ext}^1(M,N)$. *We define* $\Phi_{\mathcal{P}} : \mathcal{E}(M,N) \to \operatorname{Ext}^1(M,N)$ *by* $\Phi_{\mathcal{P}}[\zeta] = \overline{b}$.

To save space, we'll just denote $\Phi_{\mathcal{P}}$ by Φ. There is some homological algebra behind the scenes which says that the choice of $\mathcal{P}$ doesn't really matter, but we won't concern ourselves with that.

Proposition 3.1. Φ *is an isomorphism* $\mathcal{E}(M,N) \to \operatorname{Ext}^1(M,N)$.

We prove the following four statements, in this order.

1. Φ does not depend on the choice of a and b.
2. If $[\zeta] = [\zeta']$, then $\Phi[\zeta] = \Phi[\zeta']$.
3. Φ is a group homomorphism.
4. Φ is bijective.

Proposition 3.2. Φ *does not depend on the choice of* a *and* b.

Proof. Suppose that when computing $\Phi[\zeta]$, we choose $a_1 : P_0 \to E$ and $b_1 : P_1 \to E$. Then we recompute, and choose different morphisms $a_2 : P_0 \to E$ and $b_2 : P_1 \to E$. We need to verify that $\overline{b_1} = \overline{b_2}$ in $\operatorname{Ext}^1(M,N)$. That is, we need to show that $b_2 - b_1 \in \operatorname{im} f^*$.

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & & {\scriptstyle b_1}\downarrow\downarrow{\scriptstyle b_2} & & {\scriptstyle a_1}\downarrow\downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \zeta & 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \end{array}$$

Since $ta_1 = ta_2 = g$, we have $t(a_2 - a_1) = 0$. Thus $a_2 - a_1 : P_0 \to E$ has image contained in $\ker t = \operatorname{im} s$. Then by projectivity of P_0, there exists $q : P_0 \to N$ making the following diagram commute.

$$\begin{array}{ccccc} & & P_0 & & \\ & {\scriptstyle q}\swarrow & \downarrow{\scriptstyle a_2 - a_1} & & \\ N & \xrightarrow[s]{} & \operatorname{im} s & \longrightarrow & 0 \end{array}$$

Then

$$sqf = (a_2 - a_1)f = a_2 f - a_1 f = sb_2 - sb_1 = s(b_2 - b_1)$$

By injectivity of s, this implies $qf = b_2 - b_1$, that is, $f^*q = b_2 - b_1$. □

Proposition 3.3. *If* $[\zeta] = [\zeta']$, *then* $\Phi[\zeta] = \Phi[\zeta']$.

Proof. Let ζ, ζ' be two equivalent extensions of M by N (i.e. $[\zeta] = [\zeta']$).

$$\begin{array}{lccccccccc} \zeta & 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \zeta' & 0 & \longrightarrow & N & \xrightarrow{s' = \theta s} & E' & \xrightarrow{t' = t\theta^{-1}} & M & \longrightarrow & 0 \end{array}$$

Let $a, a' : P_0 \to E$ and $b, b' : P_1 \to N$ be the morphisms constructed for $\Phi[\zeta]$ and $\Phi[\zeta']$ respectively.

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \zeta & & 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \end{array}$$

By part (1), we can choose a' to be any morphism making the following diagram commute.

$$\begin{array}{ccccc} & & P_0 & & \\ & \swarrow{\scriptstyle a'} & \downarrow{\scriptstyle g} & & \\ E' & \underset{t'=t\theta^{-1}}{\twoheadrightarrow} & M & \longrightarrow & 0 \end{array}$$

In particular, we can choose $a' = \theta a$, since then the diagram commutes, as demonstrated by the following calculation.

$$t'a' = t\theta^{-1}\theta a = ta = g$$

($ta = g$ by the original diagram for ζ.) We can also choose b' to be any morphism making the following diagram commute.

$$\begin{array}{ccccc} & & P_1 & & \\ & \swarrow{\scriptstyle b'} & \downarrow{\scriptstyle a'f=\theta af} & & \\ N & \underset{s'=\theta s}{\twoheadrightarrow} & \operatorname{im} s' & \longrightarrow & 0 \end{array}$$

In particular, we can choose $b' = b$, since then the diagram commutes, as demonstrated by the following calculation.

$$s'b' = \theta sb = \theta af$$

($sb = af$ by the original diagram for ζ.) Thus $\Phi[\zeta] = \bar{b}$ and $\Phi[\zeta'] = \bar{b}$. □

Proposition 3.4. Φ *is a group homomorphism.*

Proof. Let $[\zeta], [\zeta'] \in \mathcal{E}(M, N)$. We need to show that $\Phi[\zeta + \zeta'] = \Phi[\zeta] + \Phi[\zeta']$. Choose representatives ζ, ζ'.

$$\begin{array}{ccccccccccc} \zeta & & 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \\ \\ \zeta' & & 0 & \longrightarrow & N & \xrightarrow{s'} & E' & \xrightarrow{t'} & M & \longrightarrow & 0 \end{array}$$

Then we let

$$\begin{aligned} E'' &= \{(x, x') \in E \oplus E' : t(x) = t'(x')\} \\ D'' &= \{(s(n), -s'(n)) : n \in N\} \\ F &= E''/D'' \end{aligned}$$

and we have a representative of $\zeta + \zeta'$.

$$\zeta \qquad 0 \longrightarrow N \xrightarrow{s''} F \xrightarrow{t''} M \longrightarrow 0$$

where $s''(n) = \overline{(s(n), 0)}$ and $t''\overline{(x, x')} = t(x)$. Let $a, a' : P_0 \to E$ and $b, b' : P_1 \to N$ be morphisms constructed to compute $\Phi[\zeta], \Phi[\zeta']$ respectively.

$$\begin{array}{ccccc} & & P_0 & & \\ & \overset{a}{\swarrow} & \downarrow g & & \\ E & \xrightarrow[t]{} & M & \xrightarrow{0} & 0 \end{array} \qquad \begin{array}{ccccc} & & P_0 & & \\ & \overset{a'}{\swarrow} & \downarrow g & & \\ E' & \xrightarrow[t']{} & M & \longrightarrow & 0 \end{array}$$

Then we define $a'' : P_0 \to F$ by $a''(p) = \overline{(a(p), a'(p))}$. Notice that this lies in F because $t'a'(p) = ta(p) = g(p)$ by the commutative triangles above. Then we have $t''a''(p) = ta(p) = g(p)$, so the following diagram also commutes.

$$\begin{array}{ccccc} & & P_0 & & \\ & \overset{a''}{\swarrow} & \downarrow g & & \\ F & \xrightarrow[t'']{} & M & \xrightarrow{0} & 0 \end{array}$$

By construction of b, b', we also have commutative diagrams

$$\begin{array}{ccccc} & & P_1 & & \\ & \overset{b}{\swarrow} & \downarrow af & & \\ N & \xrightarrow[s]{} & \operatorname{im} s & \xrightarrow{0} & 0 \end{array} \qquad \begin{array}{ccccc} & & P_1 & & \\ & \overset{b'}{\swarrow} & \downarrow a'f & & \\ N & \xrightarrow[s']{} & \operatorname{im} s' & \longrightarrow & 0 \end{array}$$

Then we define $b'' = b + b'$, and we calculate

$$s''b''(p) = s''(b(p) + b'(p)) = \overline{(sb(p) + sb'(p), 0)} = \overline{(sb(p), s'b'(p))} = \overline{(af(p), a'f(p))} = a''f(p)$$

so the following diagram commutes.

$$\begin{array}{ccccc} & & P_1 & & \\ & \overset{b''}{\swarrow} & \downarrow a''f & & \\ N & \xrightarrow[s'']{} & \operatorname{im} s'' & \xrightarrow{0} & 0 \end{array}$$

Thus $\Phi[\zeta + \zeta'] = \overline{b''}$, by our proposition about the freedom to choose our a'', b''. Thus

$$\Phi[\zeta + \zeta'] = \overline{b''} = \overline{b + b'} = \overline{b} + \overline{b'} = \Phi[\zeta] + \Phi[\zeta']$$

□

Proposition 3.5. Φ *is bijective.*

Proof. We define an inverse mapping. Given $\overline{b} \in \operatorname{Ext}^1(M, N)$, choose any representative b, which is a morphism $P_1 \to N$. Then let E be the pushout of b and f (see Exercise 1.9 of Schiffer). Namely,

$$E = (P_0 \oplus N)/\{(f(x), -b(x)) : x \in P_1\}$$

By Exercise 1.9, we then have a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \end{array}$$

where $s(n) = \overline{(0,n)}$ and $a(p) = \overline{(p,0)}$ and $t\overline{(p,n)} = g(p)$. Then take $[\zeta] \in \mathcal{E}(M,N)$ represented by the exact sequence on the bottom. This gives us an assignment $\Psi : \mathrm{Ext}^1(M,N) \to \mathcal{E}(M,N)$.

We need to check that this doesn't depend on the choice of b. Suppose we have $b_1, b_2 : P_1 \to N$ with $\overline{b_1} = \overline{b_2}$. Let E_i be the pushout of b_i, f, with associated morphisms s_i, t_i.

$$E_i = (P_0 \oplus N)/\{f(p), -b_i(n)\} \qquad s_i(n) = \overline{(0,n)} \qquad t_i\overline{(p,n)} = g(p)$$

By definition, $\Psi(\overline{b_1})$ and $\Psi(\overline{b_2})$ are represented by the following exact sequences.

$$\begin{array}{cccccccccc} \Psi(\overline{b_1}) & 0 & \longrightarrow & N & \xrightarrow{s_1} & E_1 & \xrightarrow{t_1} & M & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \mathrm{Id}} & & & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \Psi(\overline{b_2}) & 0 & \longrightarrow & N & \xrightarrow{s_2} & E_2 & \xrightarrow{t_2} & M & \longrightarrow & 0 \end{array}$$

We need a morphism $\gamma : E_1 \to E_2$ making the diagram above commute, so that $[\Psi(\overline{b_1})] = [\Psi(\overline{b_2})]$. Because $\overline{b_1} = \overline{b_2}$, we have $b_1 - b_2 \in \mathrm{im}\, f^*$, so there exists $\beta : P_0 \to N$ with $f^*\beta = \beta f = b_1 - b_2$. Define $\gamma : E_1 \to E_2$ by $\gamma\overline{(p,n)} = \overline{(p, n + \beta(p)}$. Note that γ is well defined because it vanishes on $\{f(p), -b_1(n)\}$, by the following calculation.

$$\gamma\overline{(f(x), -b_1(x))} = \overline{(f(x), -b_1(x) + \beta f(x))} = \overline{(f(x), -b_2(x))} = 0$$

And by the following calculation, the required diagram commutes.

$$\begin{aligned} \gamma s_1(n) &= \gamma\overline{(0,n)} = \overline{(0, n + \beta(0)} = \overline{(0,n)} = s_2(n) \\ t_2\gamma\overline{(p,n)} &= t_2\overline{(p, n + \beta(p)} = g(p) = t_1\overline{(p,n)} \end{aligned}$$

The result of all of this is that we have a well defined function $\Psi : \mathrm{Ext}^1(M,N) \to \mathcal{E}(M,N)$. Finally, we claim that Ψ is an inverse to Φ. It is immediate from the definition of Ψ that $\Phi\Psi(\overline{b}) = \overline{b}$. It remains to show that $\Psi\Phi[\zeta] = [\zeta]$. Let $[\zeta]$ have representative extension

$$0 \longrightarrow N \xrightarrow{s} E \xrightarrow{t} M \longrightarrow 0$$

then $\Phi[\zeta] = \overline{b}$ fits into the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0 \end{array}$$

Then $\Psi\Phi[\zeta] = \Psi(\overline{b})$ is the pushout of b and f.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & M & \longrightarrow & 0 \\
 & & \Big\downarrow b & & \Big\downarrow a' & & \Big\downarrow \mathrm{Id} & & \\
0 & \longrightarrow & N & \xrightarrow{s'} & E' & \xrightarrow{t'} & M & \longrightarrow & 0
\end{array}
$$

Where $E' = (P_0 \oplus N)/\sim$ and $s'(n) = \overline{(0,n)}$ and $a'(p) = \overline{(p,0)}$ and $t'\overline{(p,n)} = g(p)$. We define $\gamma : P_0 \oplus N \to E$ by $\gamma(p,n) = a(p) + s(n)$. Then

$$\gamma((f(x), -b(x)) = af(x) - sb(x) = 0$$

so γ induces a morphism $E' \to E$ by $\gamma\overline{(p,n)} = a(p) + s(n)$. Furthermore, we check that the following diagram commutes, which makes γ an equivalence between $[\zeta]$ and $\Psi\Phi[\zeta]$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{s'} & E' & \xrightarrow{t'} & M & \longrightarrow & 0 \\
 & & \Big\downarrow \mathrm{Id} & & \Big\downarrow \gamma & & \Big\downarrow \mathrm{Id} & & \\
0 & \longrightarrow & N & \xrightarrow{s} & E & \xrightarrow{t} & M & \longrightarrow & 0
\end{array}
$$

$$
\begin{aligned}
\gamma s'(n) &= \gamma\overline{(0,n)} = s(n) \\
t\gamma\overline{(p,n)} &= ta(p) + ts(n) = ta(p) = g(p) = t'\overline{(p,n)}
\end{aligned}
$$

Thus $\Phi\Psi$ and $\Psi\Phi$ are the respective identites, so Φ is a bijection. □

This concludes the proof that Φ is an isomorphism of abelian groups.

References

[1] Ralf Schiffler. Quiver representations, 2014.

[2] Charles A. Weibel. An introduction to homological algebra, 1994.